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Logical Foundations of Set Theory and Mathematics

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For much of the twentieth century the philosophy of mathematics centered around studies in the foundations of mathematics and these typically concentrated on set theory, arithmetic and the theory of real numbers. They also typically involved work in formal logic. Thus it is natural enough to think that we know what we should be talking about when given the above title. But do we really? To make sure that we do, we had better start by considering what it means to talk of foundations in general and logical foundations in particular, especially in the context of mathematics. We also have to find out how set theory gets into the act.

1 Foundations and Logical Foundations

> I was especially pleased with mathematics because of the certainty and self evidence of its proofs; but I did not yet see its true usefulness, and thinking that is was good only for the mechanical arts, I was astonished that nothing more noble had been built on so firm and solid a foundation. On the other hand I compared the ethical writings of the ancient pagans to very superb and magnificent palaces built only on mud and sand . . . when it came to the other branches of learning, since they took their cardinal principles from philosophy, I judged that nothing solid could have been built on so insecure a foundation. (Descartes 1960: 7–8)

The metaphor of building a dwelling on new, secure foundations pervades Descartes two most popular works, *Discourse on Method and Meditations*, and is one of the ways in which he influenced subsequent developments in philosophy. It was only in the second half of the twentieth century that this quest for foundations ceased to dominate philosophical discourse, although it remains a persistent theme, especially within the philosophy of mathematics, although even here it is increasingly being challenged.

When Descartes talked about foundations his conception of how to find them was influenced by what he conceived to have been the ancient mathematical method of analysis. Citing Pappus, he saw the method of analysis as a procedure for working back to the first principles upon which any putative item of knowledge would be based. The passage from Pappus in which analysis is described is as follows:

> in analysis we assume that which is sought as if it were (already) done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter and so on, until, by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being a solution backwards.
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> But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting than one with another, we arrive finally at the construction which was sought, and this we call synthesis. (Editor's note in Euclid 1926: vol. I, pp. 138–9)

It is thus by analysis that we get back to first principles, but they are only shown to be adequate as first principles if the corresponding synthesis can be completed, that is if we can show that they provide a basis from which the knowledge to be grounded can be deduced.

In Descartes' hands the analysis which provides a grounding for knowledge involves revealing the complexity of what is to be known by taking it apart into its simpler components. This may involve analysis both of objects and of concepts. Indeed terms for objects will have to be redefined by reference to the way the objects are constructed. Cartesian foundationalism was inseparable from its mechanistic reductivism. An adequate foundation is then also an ontological foundation; it tells us what our knowledge is really knowledge about. It is also epistemological in the sense of showing that and how our knowledge claims are justified. If in addition one believes that it is the ability to provide a justification that constitutes possession of knowledge, then the analysis–synthesis circuit is also a route to knowledge acquisition and analysis is a method of discovery, as Descartes himself claimed.

Descartes was not, however, looking for logical foundations. A logical foundation would be a set of first principles from which one can, using definitions and logically valid deductive arguments provide proofs for the desired knowledge claims. Descartes had a low opinion of logic, which in the seventeenth century encompassed little more than the theory of syllogisms. He wanted his foundations to provide a bedrock on which to build an edifice of human knowledge. The building would be raised by a process of deductive synthesis from principles which could 'clearly and distinctly perceived to be true' by a sequence of steps each of which was accompanied by the same sense of self-evident correctness. These steps are neither confined to the rules of any formal logic nor would logical validity automatically be sufficient for the kind of support required. A deduction must reveal what it is in virtue of which the deduced statement is true. A logically valid indirect proof using *reductio ad absurdum* frequently does not do this, it merely shows why the opposite of what is to be proved cannot be true. Moreover, if analysis involves the analysis of objects, the reversing synthesis involves the construction of objects. Knowledge of objects is grounded in their method of construction, a kind of construction altogether different from the construction of concepts using definition by genus and differentia (man is a rational animal, for example) which was the form well-suited to syllogistic reasoning.

2 Foundations for Mathematics

Why should mathematics be thought to need any foundation? This after all was the discipline which inspired Descartes, providing him with a paradigm of a structure solidly erected on firm foundations. The mathematical work central to the formation of this paradigm was Euclid's *Elements of Geometry*. There the foundations, namely, axioms, postulates and definitions, are laid out at the beginning of each book, and step by step a body of knowledge is erected through the proof of theorems, later theorems building on results established earlier. But, as a glance at Descartes' own treatise on geometry (Descartes 1925) quickly confirms, the mathematics of the seventeenth century was already moving well beyond the confines of Euclidean geometry. Newton's *Principia* (Newton 1999), written in 1686, strikingly confirms both the hold of the Euclidean paradigm as a paradigm for organizing a body of knowledge, and the extent to which mathematics and its methods have moved away from classical geometry. Newton proceeds by presenting axioms and definitions. His laws of motion are presented as axioms. His text consists of propositions proved on their basis but the means of proof introduce mathematical methods unknown to Euclid. Like Descartes Newton uses algebraic methods and builds an understanding of 'complex' motions on the basis of their composition from 'simpler' motions. This is what Kant (1996) would later call reasoning from the construction of concepts, reasoning which he contrasted with logical reasoning (reasoning from concepts). Kant saw the distinctive power of mathematics as deriving from the fact that it employs this form of constructive reasoning, reasoning grounded in the way its objects are constructed in pure intuition. Reasoning from concepts according to the laws of syllogistic logic could establish analytic truths (those based on the analysis of concepts), whereas mathematical reasoning from the construction of concepts establishes synthetic *a priori* truths.

In addition to using algebraic methods Newton introduced the language of fluxions in the process of developing the techniques which were to become 'infinitesimal calculus.' The soundness of proofs constructed by these means was quickly challenged (by Berkeley (1992) and others). The methods used seemed to many to be inherently insecure because they involved trying to treat continuous magnitudes as if they could be made up of infinitely many discrete parts. This is spite of the fact that Zeno's paradoxes (discussed by Aristotle (1996: 238b23–240b8), and used by him as a basis for insisting on an absolute distinction between discrete and continuous magnitudes) and other well-known paradoxes of the infinite suggested that such moves would lead to inconsistencies and contradictions. The mathematics that, as its use proliferated, came more and more urgently to seem to be in need of 'foundations' – solid construction on a secure base – was that of analytic geometry and the methods of infinitesimal calculus.

The challenge that infinitesimals pose to a foundationalism centered on the idea of knowledge based on methods of construction is that, even supposing there are infinitesimally small limits of division (analysis) of a continuous line, the reverse synthetic process can never be humanly completed – it would be an infinite process. It would seem to require an infinite mind to understand an infinitely complex whole on the basis of its parts. Both Kant (1996: 531) and subsequently Cantor (see Dauben 1979: 130–1) firmly declared the idea that analysis should reveal infinite complexity – structure all the way down – to be absurd. It is absurd to the extent that it violates a core principle on which the Cartesian foundational program was mounted – understanding is grounded in methods of construction. If we cannot locate simple parts at the end of a finite analysis, we humans will never reach a foundation on which to begin building. The challenge to provide a foundation for the new, infinitistic mathematics, was thus to find a way round this problem.

3 Mathematics and Set Theory

In Descartes' geometry, as also in Kant's treatment of mathematics, the problem noted above is finessed in the following way for the case of continua. They think of continuous magnitudes as constructed objects by invoking the concept of continuous action (motion). A line is constructed as the continuous motion of a point, which moves according to a law given in the form of an algebraic expression. This law expresses a complex ratio of distances from given, fixed lines (axes) whose value is constantly expressed by the moving point, and which is thus exhibited by any and every point on the constructed line. To study a curve through its algebraic characterization is then to learn about it on the basis of its method of construction, which is not a building up of discrete parts, but a continuous generation of a continuous whole.

This is a viable position as long as it is possible to think, as had been done since Aristotle's discussion of Zeno's paradoxes, that there are two irreducibly distinct kinds of whole – continuous and discrete. Europe inherited from the ancient Greeks the view that mathematics has two distinct branches – geometry, the science of continuous magnitudes, and arithmetic, the science of discrete magnitudes. Discrete magnitudes are aggregates of parts (elements); they are formed by heaping together a number of discrete items and are thus said to be 'wholes given after their parts.' A continuous whole, on the other hand, can be divided without limit and can be divided anywhere; its parts are 'created' by division which is a process of delimiting the boundaries of a part. Thus a part here is always essentially a part *of* the whole from which it is marked off and it is for this reason that continuous wholes are said to be 'wholes given before their parts.' Furthermore, because a continuous whole can be divided without limit, it potentially contains infinitely many parts. The point of distinguishing firmly between wholes given before and wholes given after their parts was to underscore the point that one cannot, on pain of contradiction, think of a continuous whole as something constructed out of the infinitely many parts it potentially contains; these parts cannot be treated as independently given discrete parts to be heaped into an aggregate.

The position taken by Kant and Descartes proved unstable for two reasons. First it was criticized for relying on the concept of motion, which, being drawn from mechanics, was unsuitable for use in thinking through the foundations of pure mathematics. Second, because it appears to place restrictions on the possible objects of mathematical study which mathematicians themselves saw no reason to recognize. It is possible to write (construct) algebraic expressions which don't correspond to any continuous or even drawable curve. What reason could be given for ruling that the complex relationships expressed in such equations should not be legitimate objects of mathematical investigation? In the eighteenth century mathematicians such as Euler and D'Alembert argued over what was to count as a function. In the end the notion of a function was liberalized in such a way that any collection of points in the plane could count as the graph of a function, and any method of calculating a real number as value for other real numbers as arguments would count as a function.

In many ways this simply reflects recognition that the move of introducing algebraic methods into geometry, of which Descartes' work was a part, and the introduction of Cartesian coordinates presupposes that each point in the Euclidean plane can be indexed by a pair of numbers, its coordinates. This in turn presupposes that a continuous line or plane can be represented by a set of numbers, or of pairs of numbers. Thus one must after all be able to view a continuum as composed of infinitely many points, in spite of the well-known contradictions arising from the supposition that one can add dimensionless points, items having length zero, together in such a way that they make up a continuous line having a positive length. The move thus involves unification of two opposed ways of thinking about part and wholes and their associated concepts of magnitude. The challenge was to find a way of doing this while avoiding the known and very real hazard of ending up with an inconsistent theory. Modern set theory proposes a solution, but without, as we shall see below, solving all the puzzles.

Mathematicians were thus firmly pushed in the direction of thinking of the Euclidean plane as an aggregate of points, if not as an aggregate constructed from points. The direction taken by Hilbert, Cantor and others was not to think about how to build up a continuum out of points, but to try to state the conditions which would have to be satisfied by a given collection of infinitely many points for them to count as constituting the points of a Euclidean plane, or a continuous line. Similarly instead of thinking about functions by starting from the lines which are their graphs, a (real valued) function of a single real variable is to be thought of as the set of ordered pairs which would be the coordinates of the points on its graph. One can then investigate what characteristics this set must possess if the function is to be continuous at a given point, differentiable at that point, and so on. Indeed Hilbert (1971) provided a new axiomatization of geometry along these lines and then proved that the real numbers could be used to 'construct' a structure (model) in which the axioms were satisfied. This appears to effect a reduction of geometry to the study of sets of points and their possible structures in conjunction with the study of real numbers.

But what is a real number? How are the real numbers defined? By making use of the concepts ordered pair, and infinite sequence mathematicians such as Cantor and Dedekind showed that one could start from the natural numbers – 0,1,2, . . . – to define the integers (negative and positive whole numbers) as ordered pairs of natural numbers where, for example, (1,2) represents 1–2, that is –1, and (2,1) represents 2–1, that is 1. Ordered pairs of integers represent the rational numbers, (1,4) is 1/4, etc. Real numbers can be defined as infinite convergent sequences (Cauchy sequences) of rational numbers. (A sequence of rational numbers is convergent if after some point the difference between successive terms gets smaller and smaller, as in 1, 1/2, 1/4, 1/8, . . .) In each case it has to be shown that the representatives have all the properties required of the numbers they are to represent. This is done by providing an axiomatic characterization of the structure required and then showing that these entities and operations defined over them can be shown to satisfy the axioms.

These moves have three possible philosophic interpretations. One (the logisist) says that the definitions show what the different kinds of numbers are and thus we have an ontological reduction of integers, rational and real numbers to natural numbers. Another (the formalist) says that these constructions prove the consistency of the axioms for integers, rational numbers and real numbers, relative to those for the natural numbers and whatever is needed for the constructions in terms of ordered pairs and infinite sequences. A third (the intuitionist) says that because the real numbers are defined as infinite, incompletable sequences, our reasoning about them has to proceed in a different way that our reasoning about the integers or rational numbers, assertion about real numbers cannot be presumed to obey the law of excluded middle. Intuitionists and constructivists resist assimilations of mathematical reasoning to logical reasoning along with any presumption that the infinite can be treated by analogy with the finite.

If the reduction could continue and the natural numbers could themselves be defined in terms of sets, then it would seem that one might be able to claim that set theory provides the ultimate foundation for mathematics. All the objects seem to be definable as sets and so in principle all theoretical results should be translatable, in principle into language which talks only about sets and operations on sets. The Bourbaki program, carried out by a group of French mathematicians, shows that this really is possible for large areas of mathematics.

The step that is made in the development of modern set theory, which allows the above constructions and allows it to accommodate aspects of the theory of both discrete and continuous wholes, wholes given before and whole given after their parts, is the distinction between set membership and set inclusion. The relationship between a set and its members, corresponds to that between a discrete whole (aggregate) and its parts and the relationship between a set and its subsets has to take over the work done by the relationship between whole given before its parts and those parts.

Sets are assumed to be identical if and only if they have the same members, so in this sense sets are defined by their members. Moreover, since the subset relationship can be defined in terms of the membership relation (A is a subset of B if and only all members of A are members of B) the barrier between these two ways of thinking about wholes and parts becomes permeable. In principle all sets are regarded as discrete wholes, even though some are infinite. However, it is also assumed that a subset of a given set A can be defined as the set of all elements of A having property P. This way of defining sets makes them subsets *of* a given set, that is parts given after the whole. It is further assumed that for any set A there is a set, the power set of A, containing as its elements all and only subsets of A. The barrier between the theory of discrete and continuous wholes, wholes given before, and those given after their parts is transformed into a double gulf (1) between finite and infinite sets and (2) between an infinite set and its power set – the set of all its subsets. The power sets of infinite sets are resistant to being treated as discrete wholes – things to which one might put a number in the same sense in which one can put a number to a finite set. This resistance is reflected in the independence of Cantor's Continuum Hypothesis from the remaining axioms of ZF set theory. (This hypothesis says that the cardinal number of the set of all subsets of the natural numbers is the next infinite cardinal number after that of the set of natural numbers. Cantor had already proved that the cardinal number of the set of real numbers is the same as the cardinal number of the set of all subsets of the natural numbers.)

4 Sets, Classes, and Logic

So how does enquiry into the foundations of mathematics become a quest for logical foundations? By relating sets to classes and in this way making set constructions the product of corresponding logical operations for defining predicates. Then the way is cleared for losing the distinction between the synthesis which is logical deduction from first principles and the synthesis which is building up from simple component parts, and hence also the distinction between logical analysis (analysis of concepts) and analysis of objects. This will work if sets or classes are objects which can be constructed by logical operations on their corresponding concepts, but would not be possible without extending logic to cover relations and functions, as well as concepts, and the various operations used by mathematicians to define these. Accomplishing this task was Frege's major achievement.

Frege (1950) aimed to show that arithmetic is a body of analytic truths; that it really is a part of logic, in his new extended sense of logic. This includes the claims that classes are logical objects, that numbers are classes and that the application of any arithmetical truth is a matter of logical deduction. If Frege had succeeded he would thus have explained the universal applicability of arithmetic at the same time as providing it with a foundation in logic and the theory of classes.

The notion of set, or class, invoked in an informal way by Cantor and other mathematicians, already had a history in logic and attempts to introduce algebraic methods into logic, from Leibniz to Venn, De Morgan, and Boole. In traditional logic a class is the extension of a term – the collection of objects of which that term can be correctly predicated. Classes are thus wholes to which the theory of discrete, rather than continuous magnitude would apply.

The first thing that Frege needed to do was to introduce into logic a reflection of the distinction between the membership and subset relations. In Aristotelian logic this was not marked because singular statements, such as 'Aristotle was bald' were, for the purposes of syllogistic logic, treated as universal sentences, that is by analogy with 'All Greek males are bald.' Both of these would have been assigned the form 'S a P' and would then be viewed as expressing either an intensional relation (the predicate P is included in the concept of the subject S) or an extensional relation (the extension of the subject term S is included in the extension of the predicate term P). Frege on the other hand insisted on the distinction between object and concept as a logical distinction and one that should be reflected in logical notation. Objects have to be reflected at the logical level if the application of numbers is to be a logical operation, for it is objects that are counted and it is objects that are formed into sets.

The logic we have inherited from Frege, via Russell and others, thus starts from the singular sentence, P(a) which corresponds to the set theoretic form '$a \in \{x : Px\}$.' The universal then has the form '$\forall x(S(x) \in P(x))$' which in turn can be used to define the subset relation; $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$. Frege also argued that set theory had to be based in logic if it was to hope to account for numbers and our use of them. The idea of a set as an aggregate of objects runs into problems trying to account for the bases of the system of natural numbers – 0 and 1. How can there be a heap containing no objects? Moreover what is the difference between a heap containing a single object and just a single object. Frege's insistence that sets should be thought of as classes, the extensions of concepts, avoids these puzzles. It is easy to define a concept ('is a round square' or 'x ≠ x' for example) under which no object can possibly fall, and which hence has an empty extension. So 0 is the number of the concept 'x ≠ x.' Similarly there can be concepts under which only one object falls ('0 = x,' for example) whose extensions contain a single object. So 1 is the number of the concept '0 = x.' Frege thus asserted that a statement in which a number is applied is a statement about a concept; it says how many things fall under it. But he also insisted that numbers are themselves objects which can be grouped into classes. He ends up defining numbers as classes, saying that for any concept F, the number of Fs is the class of classes which are equinumerous with the class of Fs. So, for example 1 becomes the class of all classes equinumerous with the class of things identical to 0.

With the numbers so defined Frege shows, using only his logical principles and definitions, that they will satisfy the axioms for the natural numbers, given earlier by Peano. This would justify his claim that the truths of arithmetic are really logical truths, expressible using only logical concepts such as identity, object, concept, and class together with logical operations, such as negation, conjunction, and the formation of universal and existential generalizations (expressed with his newly introduced quantifier/bound variable notation). Unfortunately, as is well-known, Frege's logic was shown by Russell to be inconsistent; it permits the existence of the class of all classes which do not belong to themselves and if this class either belongs or does not belong to itself, a contradiction results.

Russell's response (Whitehead and Russell 1910–13) was to place restrictions on the predicates which could be thought to determine classes. His vicious circle principle, used in developing the ramified theory of types, bans classes from being defined and formed by reference to more encompassing classes to which they would belong. So, for example, no class can be defined by referring to the totality of all classes, since it would itself belong to that totality. This principle insists that the 'parts,' or members, of a discrete whole must be definable independently of that whole. In addition Russell insists that classes are basically logical fictions, not genuine objects. In other words, statements about a class should in principle be expressible as statements about the members of that class. This would not be possible if the vicious circle principle were violated. His image is then very reductivistically foundational, with a vision of a universe of classes which can be built up successively from a given stock of individuals, and where the whole superstructure could in principle be shown to provide only a shorthand for making complex descriptions of that universe of individuals. This vision had great appeal to empiricists since it appeared to obviate the need to postulate the existence of any abstract objects in order to account for mathematical knowledge and its wide applicability.

The problem is that, as Russell himself was forced to recognize, this does not yield a theory of classes which meets mathematicians' requirements. If we remember that what mathematicians required was a unification of the theory of wholes given before their parts with that of wholes given after their parts, we can understand why Russell's complex system, although much richer than anything achievable with traditional logic, will not serve, for it is constrained to a theory of wholes given after their independently specifiable parts and replaces set construction by logical construction of their defining predicates.

In order to have a theory rich enough to develop mathematics Russell had to add two specific axioms – Infinity, which says that there are infinitely many individuals, and Reduction, which basically allows the existence of all subclassess of a given class, no matter how defined, to be collected into a class. Both of these are existence axioms and cannot easily be claimed to be logical truths. Moreover their use raises once again the problem of consistency – how could one be sure that tacking these two axioms onto the system will not render it inconsistent?

An alternative response to the problems with Frege's logic was to axiomatize the theory of sets and then think about how to prove the axiomatized theory consistent. The axiomatization now regarded as standard is based on those of Zermelo and Fraenkel (hence called ZF). It includes operations for building up sets member by member, but also for an infinite set and for using predicates to mark off the subsets of an already given set. The totality of subsets of a given set is asserted to exist without any restriction which says that these have to be definable as the extensions of predicates. Moreover, in many cases an additional axiom, the Axiom of Choice is added, and this explicitly asserts the existence of sets as aggregates of objects for which there may well be no such definition. Gödel (1938) showed that it is possible to provide a model for the ZF axioms by restricting sets to those which are definable (the constructive universe). In this universe the axiom of choice and Cantor's continuum hypothesis would be true. However he and others have also argued that this universe is too restrictive for mathematical purposes. Subsequently Cohen (1966) proved that both the axioms of choice and the continuum hypothesis are independent of the remaining axioms of ZF set theory. This means that the basic ZF axioms remain neutral on whether set construction is reducible to logical construction, but to the extent that mathematics seems to require use of the axiom of choice and to presume a universe containing nonconstructible sets, this reductive restriction is rejected.

The resulting relation between logic and set theory is complex. It is certainly not a matter of one providing a foundation for the other. ZF set theory is written in the language of classical first-order predicate logic and any results proved about theories written in such a language apply to set theory. Some of those results, however, are proved using set theory, since the semantic approach to the study of predicate logic, relies on the concept of a model, and models are defined as structured sets. Results about models are then proved in set theory. So there is a complex, symbiotic relation between axiomatic set theory and predicate logic.

Hilbert's (formalist) program was to develop finitary methods for theorizing about formally expressed axiomatic theories with the aim of proving whether or not they are consistent. The idea was that if it could be proved using only finitary methods that a theory of infinite sets was consistent (that no formal contradiction could be proved from the axioms) then it would be safe to use. Again this is a way of seeking to use a constructive base to legitimize something which goes beyond it.

Gödel (1962) contains his famous incompleteness results. His first incompleteness theorem showed that any consistent formal system capable of expressing arithmetic would contain undecidable arithmetic sentences. On the assumption that any statement about numbers is either true or false, this would imply that there would always be some arithmetic truth that could not be proved in the particular formal system in question. This creates a problem for the logicist who wants to say that every arithmetic truth is a logical truth. It either has to be allowed that no formal system captures the notion of logical truth, or that the logicist claim is false, or that not every statement about numbers is determinately true of false. His second incompleteness result shows that the consistency of such a system cannot be proved by means formalizable within the system, which demonstrates that Hilbert's program for providing an ultimate consistency proof for infinitary methods by finitary means cannot be realized.

Where did this leave foundational programs? Although Gödel's results undercut the philosophical rationale for both logicist and formalist programs, foundational studies had taken on a life of their own. New branches of mathematics, and new ways of studying logics had been developed. There were plenty of things to be discovered about these new domains and work in all these areas for a while continued to fall under the title studies in the foundations of mathematics. Philosophers too needed to learn from the technical results to try to decipher their philosophical significance. The idea that mathematics has a foundation in logic could still be pursued by debating the boundaries of logic and the way in which a reduction to logic might be effected. However, that particular convergence of mathematics, set theory and logic required to reduce the construction of mathematical objects to logical construction (definition of predicates), which was central to the plausibility of the claim that mathematics could be provided with a foundation in logic, proved to be relatively short lived. By the late twentieth century logic, set theory and mathematics were developing on independent tracks, interacting in complex ways, but none serving as a bedrock on which to raise the others.

The weaker claim that any branch of mathematics can be given a logical foundation, by being written as an axiomatized theory in the language of first order logic, leads to a different way of saying that all mathematical truths are logical truths. One can then say that what mathematicians prove are logical truths of the form 'If P then A,' where P is some finite conjunction of axioms. If the axioms are inconsistent, all such statements are still be logical truths, given the materiality of the conditional in classical first-order logic. Unfortunately this gives a much too simplistic picture of mathematical practice. Take for example, Wyle's proof of Fermat's last theorem. This appeals to results in many branches of mathematics other than arithmetic. To even begin to represent his proof as establishing a logical connection one would include the axiomatizations of all these other bits of mathematics, and give a logical representation of the process of applying the results from one mathematical domain in another. Thanks to the work of the Bourbaki group in showing how to do mathematics within the framework of set theory, one might say that in principle this could be done within set theory; but others would question whether such a thing (a full formal proof) would be able to serve the functions of a proof – convincing people by helping them understand why the conclusion is true.

The focus of foundational studies was set in the nineteenth century at a time when it seemed that numbers of various kinds were the fundamental objects of mathematical investigation. In the twentieth century mathematics seemed to be equally concerned with the investigation of structures and procedures. Structures can be characterized without saying how they can be built from objects. They can be characterized on the basis of the kinds of transformations under which they are preserved. This idea gave rise to a rival foundational bid from category theory, where objects are complex wholes given before their parts and internal structure is revealed through a study of the way they relate to other objects of their kind (category) through structure preserving mappings (morphisms).

The study of finitary procedures led to the theory of recursive and computable functions and to the developments of electronic computers. The extensive use and deployment of these computers has in turn been instrumental in undermining some of the presumptions which made foundational programs seem plausible. In particular development of the study of fractals, and complex systems, coupled with earlier results in nonstandard analysis, show that there is no more risk of contradiction associated with infinitesimals and the idea of structure all the way down, than with infinitely large sets.

Attempts to make computers into expert systems have stimulated the study of alternative logics, some of which (particularly non-monotonic and fuzzy logics) depart radically from the systems developed by Frege and Russell. In addition uses such as computer modeling have meant that there is continued interest in mathematics developed by constructivists, those who resisted both the move to reduce mathematics to logic and the use of infinitistic methods. Since computer memories are decidedly finite, computer representations of the continuous have to be based on finitary, approximative methods.

So we are once again in a context where it is not at all clear what a logical foundation for mathematics would look like, nor is it clear that logic is the place to look for foundations or even that foundations are what we need to be looking for.

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